

Fermion Particle Production in Dynamical Casimir Effect in a Three Dimensional Box

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In this paper we investigate the problem of fermion creation inside a three dimensional box. We present an appropriate wave function which satisfies the Dirac equation in this geometry with MIT bag model boundary condition. We consider walls of the box to have dynamic and introduce the time evolution of the quantized field by expanding it over the instantaneous basis. We explain how we can obtain the average number of particles created. In this regard we find the Bogliubove coefficients. We consider an oscillation and determine the coupling conditions between different modes that can be satisfied depending on the cavity spectrum. Assuming the parametric resonance case we obtain an expression for the mean number of created fermions in each mode of an oscillation and their dynamical Casimir energy.

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I. INTRODUCTION

The Casimir effect is regarded as one of the most striking manifestation of vacuum fluctuations in quantum field theory. The presence of reflecting boundaries alters the zero-point modes of a quantized field, and results in the shifts in the vacuum expectation values of quantities quadratic in the field, such as the energy density and stresses. In particular, vacuum forces arise acting on constraining boundaries. The particular features of these forces depend on the nature of the quantum field, the type of spacetime manifold and its dimensionality, the boundary geometries and the specific boundary conditions imposed on the field. Since the original work by Casimir in 1948 [1] many theoretical and experimental works have been done on this problem (see, e.g., [2–5] and references therein). A new phenomenon, a quantum creation of particles (the dynamical Casimir effect) occurs when the geometry of the system varies in time. Theoretical investigations in this regard are for various fields, geometries, boundary conditions, and different dimensions. Here we mention just some of them: motion of a single reflecting boundary [6], the vacuum stress induced by uniform acceleration of a perfectly reflecting plane [7], a sphere expanding in the four-dimensional spacetime with constant acceleration investigated by Frolov and Serebriany [8, 9] in the perfectly reflecting case and by Frolov and Singh [10] for semi-transparent boundaries, and more general cases of motion by vibrating cavities considered on the base of various perturbation methods [11–18]. Particle creation from the quantum scalar vacuum by expanding or contracting spherical shell with Dirichlet boundary conditions is considered in [19]. In another paper the case is considered when the sphere radius performs oscillation with a small amplitude and the expression are derived for the number of created particles to the first order of the perturbation theory [20]. Consid-

ering that Creation of particles and dynamical Casimir energies of configurations depend on the nature of the particular quantum field, now in the present paper by using the result of [21] we concentrate on the creation of particles for the Dirac field. Using the Dirac field in three dimensional box we investigate the number of fermion production in this geometry with dynamical boundaries for an arbitrary motion of the walls and then for an oscillatory modulation.

In order to obtain the number of produced fermions organization of this paper is as follows, in sec. II. we present an appropriate wave function which satisfies the Dirac equation in side a box with MIT bag model boundary condition. We consider the walls of the box to have dynamic, then we introduce the time evolution of the quantized field by expanding it over the instantaneous basis. Following the steps given in [18] we arrive at an infinite set of coupled differential equations for the coefficients of the expansion. We explain how we can obtain the average number of particles created after the end of the motions. Dependence of this number to the Bogliubove coefficients leads to find the Bogliubove coefficients in sec. III. We consider an oscillation with small amplitudes of oscillations and determine the coupling conditions between different modes that can be satisfied depending on the cavity spectrum. Assuming the parametric resonance case we obtain an expression for the mean number of created fermions in each mode of an oscillation and dynamical casimir energy corresponding to the created particles.

II. EXPANSION OF DIRAC FIELD OVER THE INSTANTANEOUS BASIS

Let's have a brief review on the construction of the eigenfunctions of the Hamiltonian for the Dirac field inside a box [21]. In order to satisfy the purpose of appropriate confinement physically relevant boundary condition must be imposed. A proper boundary condition for the Dirac field is the MIT bag model boundary condition. This model was first considered by Bogoliubov [22] and

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later developed as the MIT bag model by Chodos et al. [23] for hadrons. It is usually said to imply that there is no flux of fermions through the boundary. However, it implies an even stronger condition that is the absolute confinement of the Dirac field (see also [24]). Considering all of the back and forth terms inside the box the most general stationary solution is

$$\Psi(x_1, x_2, x_3, t) = \psi(x_1, x_2, x_3) \exp(-iEt), \quad (1)$$

where the spatial part is

$$\begin{aligned} \psi(\mathbf{x}, t) = & f \left(\frac{\alpha}{E+m} \right) e^{i(p_1 x_1 + p_2 x_2 + p_3 x_3)} + g \left(\frac{\beta}{E+m} \right) \\ & \times e^{i(-p_1 x_1 + p_2 x_2 + p_3 x_3)} + h \left(\frac{\eta}{E+m} \right) e^{i(p_1 x_1 - p_2 x_2 + p_3 x_3)} \\ & + j \left(\frac{\mu}{E+m} \right) e^{i(p_1 x_1 + p_2 x_2 - p_3 x_3)} + k \left(\frac{\nu}{E+m} \right) \\ & \times e^{-i(-p_1 x_1 + p_2 x_2 + p_3 x_3)} + l \left(\frac{\tau}{E+m} \right) e^{-i(p_1 x_1 - p_2 x_2 + p_3 x_3)} \\ & + q \left(\frac{\chi}{E+m} \right) e^{-i(p_1 x_1 + p_2 x_2 - p_3 x_3)} + r \left(\frac{\rho}{E+m} \right) \\ & \times e^{-i(p_1 x_1 + p_2 x_2 + p_3 x_3)}. \end{aligned}$$

Here \vec{p} denotes momentum operator and $\alpha, \beta, \eta, \mu, \nu, \tau, \chi$, and ρ are general two-component spinors and coefficients f through r can be determined. Imposing the prevalent form of the MIT bag model boundary condition on the dirac field inside a cubic box of side a as follow

$$(1 \pm i n_\mu \gamma^\mu) \psi(\mathbf{x}) \Big|_{x_i = \pm a/2} = 0, \quad i = 1, 2, 3 \quad (3)$$

on the Eq. (2) yields in the quantization condition for all components of the momentum

$$p_i \cot p_i a = -m, \quad i = 1, 2, 3. \quad (4)$$

Now suppose that the distance a varies as a function of time $a(t)$. Assume that $a(t)$ has a constant initial value a and after a time ΔT the modulation stops and $a(t)$ takes its initial value. The Fourier expansion of the field for an arbitrary moment of time, in terms of creation and annihilation operators, can be written as

$$\psi(\mathbf{x}, t) = \sum_n a_n^{in} u_n(\mathbf{x}, t) + b_n^{\dagger in} v_n(\mathbf{x}, t). \quad (5)$$

We use the index “in” to mean times before modulation of the system and a_n^{in} and $b_n^{\dagger in}$ are the annihilation and creation operators correspond to the particles in the “in” region. The mode functions $u_n(x, t)$ form a complete orthonormal set of solutions of the wave equation with MIT bag model boundary conditions. For the static cavity each field mode is determined by the Eq. (4) and the mode functions $u_n(x, t)$ have the form of the static

solution introduced in Eq. (1). When the modulation begins the boundary conditions on the moving walls become time-dependent. To satisfy these time-dependent boundary condition we expand the mode functions with respect to an instantaneous basis [25]:

$$u_n(\mathbf{x}, t) = \sum_k (i\gamma^\nu \partial_\nu + m) Q_k^{(n)}(t) \varphi_k(\mathbf{x}, t). \quad (6)$$

We suppose that two-component spinors are identical for simplicity and consequently we have an overall normalization coefficient f ,

$$\begin{aligned} \varphi_k(\mathbf{x}, t) = & \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \{ e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3)} + e^{i(-k_1 x_1 + k_2 x_2 + k_3 x_3)} \\ & + e^{i(k_1 x_1 - k_2 x_2 + k_3 x_3)} + e^{i(k_1 x_1 + k_2 x_2 - k_3 x_3)} \\ & + e^{-i(-k_1 x_1 + k_2 x_2 + k_3 x_3)} + e^{-i(k_1 x_1 - k_2 x_2 + k_3 x_3)} \\ & + e^{-i(k_1 x_1 + k_2 x_2 - k_3 x_3)} + e^{-i(k_1 x_1 + k_2 x_2 + k_3 x_3)} \} \end{aligned} \quad (7)$$

where

$$u_+ = \eta, \quad u_- = \frac{\vec{\sigma} \cdot \vec{k}}{E+m} \eta. \quad (8)$$

Considering continuity of each field mode and its time derivative at $t = 0$ the initial conditions are

$$\begin{aligned} Q_k^{(n)}(0) &= f \delta_{n,k} \\ \dot{Q}_k^{(n)}(0) &= -i\omega_n f \delta_{n,k}, \end{aligned} \quad (9)$$

and due to the normalization condition [26] we have

$$f = \sqrt{\frac{(E+m)}{2EV}},$$

where V denotes volume of the box. The expansion in Eq.(6) for the field modes must be a solution of the wave equation. Inserting it in the Dirac equation and taking into account that the φ_k 's form a complete and orthogonal set of solutions of the wave equation and that they depend on t only through $a(t)$, we obtain a set of coupled equations for $Q_k^{(n)}(t)$:

$$\begin{aligned} \ddot{Q}_k^{(n)}(t) \varphi_k(\mathbf{x}, t) + \omega_k^2(t) Q_k^{(n)}(t) \varphi_k(\mathbf{x}, t) = & \quad (10) \\ - 2\dot{Q}_k^{(n)}(t) \dot{\varphi}_k(\mathbf{x}, t) - Q_k^{(n)}(t) \ddot{\varphi}_k(\mathbf{x}, t), \end{aligned}$$

with $\omega_k^2 = m^2 + |\vec{k}|^2$.

III. DERIVATION OF BOGOLIUBOV COEFFICIENTS

When the boundaries of the box return to their initial position φ_k 's are time independent and the right-hand side in Eq.(10) vanishes. We call this region of time “

out" region and we can define a new set of creation and annihilation operators. The Fourier expansion of the field after modulation is

$$\psi(\mathbf{x}, t) = \sum_n a_n^{out} u_n(\mathbf{x}, t) + b_n^{\dagger out} v_n(\mathbf{x}, t). \quad (11)$$

In this region Eq.(10) reduces to the following simple form

$$(\partial_t^2 + \omega_p^2) Q_p^{(n)}(t) = 0, \quad (12)$$

and then the solution reads

$$Q_p^{(n)}(t) = A_p^{(n)} e^{i\omega_p t} + B_p^{(n)} e^{-i\omega_p t}, \quad (13)$$

with $A_p^{(n)}$ and $B_p^{(n)}$ constant coefficients to be determined. The creation and annihilation operators for particles and anti-particles in the *in* and *out* regions obey the usual anticommutation relations. Using the Bogoliubov canonical transformation one can expand the *out* operators in terms of the *in* operators

$$\begin{aligned} a_k^{out} &= \alpha_k a_k^{in} + \beta_k b_{-k}^{\dagger in} \\ b_k^{\dagger out} &= -\beta_k^* a_k^{in} + \alpha_k^* b_{-k}^{\dagger in}. \end{aligned} \quad (14)$$

Substituting Eq. (13) in Eq. (6) and then in Eq. (11), and considering Eq. (14) we expand φ_k for out region in term of the creation and annihilation operators in the in region. After some calculation and by means of following relations that can be seen easily

$$\begin{cases} (\vec{\sigma} \cdot \vec{p}) u_- = (E - m) u_+ \\ (\vec{\sigma} \cdot \vec{p}) u_+ = (E + m) u_- \end{cases} \quad (15)$$

eventually we get

$$\alpha_p^* = - \left(\frac{mu_+ + \omega_p u_- - (E - m) u_+}{mu_- + \omega_p u_+ + (E + m) u_-} \right) B_p^{(n)}. \quad (16)$$

Using the relation between Bogoliubov coefficient for fermionic fields

$$|\alpha_p|^2 + |\beta_p|^2 = 1, \quad (17)$$

the mean number of particles produced in the mode \vec{p} through an arbitrary modulation of the single fermion mode is the average value of the number operator with respect to the initial vacuum state

$$N_p = |\beta_p|^2 = 1 - 2\omega_p^2 |B_p^{(n)}|^2. \quad (18)$$

In the presence of the boundaries, all of the components of the momentum are subject to quantization condition Eq.(4). Therefore the dynamical Casimir energy related to the particles production is given by

$$E = \sum_s \sum_p N_p \sqrt{m^2 + |\vec{p}|^2}, \quad (19)$$

where the summation index s runs over the spin states. We are not able to sum over quantized mode of a massive Dirac field given by Eq. (4) analytically. By setting $m = 0$ we concentrate on the massless Dirac field, the quantization condition Eq.(4) in this case yields the simple form of $p_i = (n_i + \frac{1}{2}) \frac{\pi}{a}$ for a mode \vec{p} and then consequently the dynamical Casimir energy for these massless case becomes

$$E = \sum_{s,n} \sum_{n_1, n_2, n_3=0}^{+\infty} (1 - 2\omega_p^2 |B_k^{(n)}|^2) \omega_p, \quad (20)$$

where $\omega_p = \sqrt{(n_1 + \frac{1}{2})^2 + (n_2 + \frac{1}{2})^2 + (n_3 + \frac{1}{2})^2}$.

Up to this point the equations are valid for an arbitrary motion of the boundaries of the box. We only assume $a(0) = a$. We are interested in the number of fermions created inside the box, so we look for harmonic oscillations of the walls which could enhance that number by means of resonance effects for some specific external frequencies. So we study the following oscillations

$$a(t) = a(1 + \varepsilon \sin(\Omega t)). \quad (21)$$

For small amplitudes of oscillations $\varepsilon \ll 1$ contenting ourselves with first order, the equations for the modes Eq.(10) takes the form [27]

$$\begin{aligned} \ddot{Q}_p^{(n)}(t) + \omega_p^2 Q_p^{(n)}(t) = & \frac{E + m}{2EV} \left[2\varepsilon \Omega \cos(\Omega t) \sum_k g_{pk} \dot{Q}_k^{(n)}(t) \right. \\ & \left. - \varepsilon \Omega^2 \sin(\Omega t) \sum_k g_{pk} Q_k^{(n)}(t) + 2\varepsilon \omega_p^2 \sin(\Omega t) Q_p^{(n)}(t) \right], \end{aligned} \quad (22)$$

where

$$g_{pk} = a(t) \int_0^{a(t)} d^3x \frac{\partial \varphi_p^*(\mathbf{x}, t)}{\partial a} \varphi_k(\mathbf{x}, t). \quad (23)$$

Since $\varepsilon \ll 1$ it is natural to assume that the solution of Eq. (22) is of the form

$$Q_p^{(n)}(t) = A_p^{(n)}(t) e^{i\omega_p t} + B_p^{(n)}(t) e^{-i\omega_p t}, \quad (24)$$

with the function $A_p^{(n)}(t)$ and $B_p^{(n)}(t)$ varying slowly with time. We insert Eq. (24) into Eq. (22) to obtain differential equations for $A_p^{(n)}(t)$ and $B_p^{(n)}(t)$. After neglecting their second derivatives and multiplying equation by

$e^{\pm i\omega t}$ we average over the fast oscillations, we have

$$\begin{aligned} \frac{dA_p^{(n)}}{d\tau} = & \frac{E+m}{2EV} \left\{ -\frac{\omega_p}{2} B_p^{(n)} \delta(2\omega_p - \Omega) \right. \\ & + \sum_k (-\omega_k + \frac{\Omega}{2}) \delta(-\omega_p - \omega_k + \Omega) \frac{\Omega}{2\omega_p} g_{pk} B_k^{(n)} \\ & + \sum_k \left[(\omega_k + \frac{\Omega}{2}) \delta(\omega_p - \omega_k - \Omega) + (\omega_k + \frac{\Omega}{2}) \delta(\omega_p - \omega_k + \Omega) \right] \\ & \times \frac{\Omega}{2\omega_p} g_{pk} A_k^{(n)} \left. \right\}, \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{dB_p^{(n)}}{d\tau} = & \frac{E+m}{2EV} \left\{ -\frac{\omega_p}{2} A_p^{(n)} \delta(2\omega_p - \Omega) \right. \\ & + \sum_k (-\omega_k + \frac{\Omega}{2}) \delta(-\omega_p - \omega_k + \Omega) \frac{\Omega}{2\omega_p} g_{pk} A_k^{(n)} \\ & + \sum_k \left[(\omega_k + \frac{\Omega}{2}) \delta(\omega_p - \omega_k - \Omega) + (\omega_k + \frac{\Omega}{2}) \delta(\omega_p - \omega_k + \Omega) \right] \\ & \times \frac{\Omega}{2\omega_p} g_{pk} B_k^{(n)} \left. \right\} \end{aligned} \quad (26)$$

where $\tau = \varepsilon t$ is a time scale. Now we shall solve equations (25) and (26). Depending on the walls frequency and the spectrum of the static cavity we have different kinds of solutions. If we consider $\Omega = 2\omega_p$ namely the frequency of the boundaries are twice the frequency of some unperturbed mode, in this condition if $\omega_k - \omega_p = \Omega$ the resonant mode p will be coupled to some other mode k . Let us suppose that coupling conditions $|\omega_p \pm \omega_k| = \Omega$ is not fulfilled. In this case and for a massless field the equations (25) and (26) reduces to

$$\frac{dA_p^{(n)}}{d\tau} = -\frac{\omega_p}{4V} B_p^{(n)}, \quad (27)$$

$$\frac{dB_p^{(n)}}{d\tau} = -\frac{\omega_p}{4V} A_p^{(n)}. \quad (28)$$

The solutions of these coupled equations must satisfy the

initial condition mentioned in Eq. (9), it reads

$$A_p^{(n)} = -\frac{\delta_{np}}{\sqrt{2V}} \sinh\left(\frac{\omega_p}{4V}\tau\right), \quad (29)$$

$$B_p^{(n)} = \frac{\delta_{np}}{\sqrt{2V}} \cosh\left(\frac{\omega_p}{4V}\tau\right). \quad (30)$$

By using Eq.(18) the average number of produced fermions in the mode p is

$$N_p = 1 - \frac{\omega_p^2}{V} \cosh^2\left(\frac{\omega_p}{4V}\tau\right), \quad (31)$$

and dynamical Casimir energy of the created fermions is

$$E = 2 \sum_{n_1, n_2, n_3=0}^{+\infty} \left(1 - \frac{\omega_p^2}{V} \cosh^2\left(\frac{\omega_p}{4V}\tau\right)\right) \omega_p. \quad (32)$$

IV. CONCLUSION

In this paper we have discussed the particle creation for a Dirac field in a three dimensional box with the MIT bag model boundary condition. We have considered all boundaries of the box to modulate during a finite time interval. We have used the Bogoluibove coefficients in order to obtain the number of fermion created during the motion. It is worth mentioning that we derived the Bogoluibove coefficients to be tetrad for the Dirac field. We have taken into account the usual parametric resonance case ($\Omega = 2\omega_p$). We have also computed the dynamical Casimir energy for this case for the mentioned modulation and the parametric resonance case.

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